Quadratic variation

Def $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ - a filtration, (A_t) - adapted to $(\mathcal{F}_t)$.

(A_t) is said to have bounded variation if $t \to (A_t)$ - right-continuous and of bounded variation.

Can define $(X\cdot A)_t(\omega) := \int_0^t X_s(\omega)\, dA_s(\omega)$ for X such that $X: [0,t]\times\mathbb{R} \longrightarrow \mathbb{R}$ is $(\mathcal{F}_t \times \mathcal{L})$ measurable $\forall t$ (progressively measurable).

But we have:

Lemma. Let $(M_t)_{t\in\mathbb{R}_+}$ be a continuous martingale of bounded variation. Then $M_t \equiv$ const a.s.

As with Brownian motion, it follows from

Theorem If M is continuous bounded martingale then it has a quadratic variation:

$$\langle M,M\rangle_t = \underset{|\Delta|\to 0}{P\text{-}\lim} \sum \left(M_{t_k} - M_{t_{k-1}}\right)^2$$

Moreover, $\langle M,M\rangle_t$ is unique continuous increasing (even bounded variation) process with $\langle M,M\rangle_0 \equiv 0$ such that $M_t^2 - \langle M,M\rangle_t$ is a martingale.

Proof (outline). Let $\Delta = \{0 = t_1, t_2, \dots\}$ be a partitition of $\mathbb{R}_+$.

$$T_t^\Delta(M) := T_t^\Delta := \sum_{i=1}^{n-1} (M_{t_{i+1}} - M_{t_i})^2 + (M_t - M_{t_n})^2, \text{ where } n = \max\{i : t_i < t\}$$

Observe: $M_t^2 - T_t^\Delta$ - continuous martingale.

$\big(E(M_t^2 - T_t^\Delta \mid \mathcal{F}_s) = E(M_t^2 - M_s^2 \mid \mathcal{F}_s) + E(M_s^2 \mid \mathcal{F}_s) - T_s^\Delta -$
$E\left((T_t^\Delta - T_s^\Delta \mid \mathcal{F}_s\right) = M_s^2 - T_s^\Delta$, since

for $t_i < s < t_{i+1}$, $E\left((M_{t_{i+1}} - M_{t_i})^2 \mid \mathcal{F}_s\right) = E\left((M_{t_{i+1}} - M_s)^2 \mid \mathcal{F}_s\right) + (M_s - M_{t_i})^2$, and if $t_i > s$, then $E\left((M_{t_{i+1}} - M_{t_i})^2 \mid \mathcal{F}_s\right) -$

$E\left(E\left((M_{t_{i+1}}-M_{t_i})^2 \mid \mathcal{F}_{t_i}\right) \mid \mathcal{F}_s\right) = E\left(M_{t_{i+1}}^2 - M_{t_i}^2 \mid \mathcal{F}_s\right)$, so

$$E\left(T_t^{\Delta} - T_s^{\Delta} \mid \mathcal{F}_s\right) = E\left(M_t^2 - M_s^2 \mid \mathcal{F}_s\right)\}.$$

Need to show. $|\Delta_n| \to 0 \Rightarrow T_t^{\Delta_n}$ converges in L^2_{loc}.

Observe: Δ, Δ' - two subdivisions, then

$$T_t^{\Delta} - T_t^{\Delta'} = \left(M_{\cancel{t}}^2 - T_t^{\Delta'}\right) - \left(M_t^2 - T_t^{\Delta}\right) - \text{martingale.}$$

and $E\left((T_t^{\Delta} - T_t^{\Delta'})^2\right) = E\left(\underbrace{T_t^{\Delta \cup \Delta'}(T^{\Delta} - T^{\Delta'})}_{\text{Operation } T_t \text{ applied to martingale } T^{\Delta}-T^{\Delta'}.}\right)$

By $(x+y)^2 \le 2(x^2+y^2)$, we have

$$T_t^{\Delta \cup \Delta'}(T^{\Delta} - T^{\Delta'}) \le 2\left(T_t^{\Delta \cup \Delta'}(T^{\Delta}) - T_t^{\Delta \cup \Delta'}(T^{\Delta'})\right)$$

So to show local L^2 convergence, it is enough to show that $E\left(T_t^{\Delta \cup \Delta'}(T^{\Delta})\right) \to 0$ as $|\Delta| + |\Delta'| \to 0$.

This is done by Doob Maximal Thm + Schwarz inequality

So we get that $\exists\, F(t)$: $\lim_{|\Delta_n| \to 0} E\left((T_t^{\Delta_n} - F(t))^2\right) = 0$ $\forall t$.

Again by Doob, it implies that $E\left(\sup_{s \le t}(T_s^{\Delta_n} - F(s))^2\right) \to 0$.

So $\exists\, \Delta_n$ - sequence of partitions, $\Delta_{n+1} \supset \Delta_n$ - refinement, such that

$\sup_{s \le t} |T_s^{\Delta_n} - F(s)| \to 0$ a.s. in P.

So, since $T_s^{\Delta_n}$ is continuous, so is $F(s)$.

$\forall s, t \in \bigcup_n \Delta_n$, $s < t \Rightarrow \exists\, N: s, t \in \Delta_N$.

Then $T_s^{\Delta_n} \le T_t^{\Delta_n} \Rightarrow F(s)$ is increasing on $\bigcup \Delta_n \Rightarrow$

by continuity, $F(s)$ is increasing on $[0,t]$.

(although, T_s^{Δ} is not always increasing).

Since $T_t^{\Delta_n} \xrightarrow{L^2} F$, and $M_t^2 - T_t^{\Delta_n}$ is a martingale, so is $M_t^2 - F$. Take $\langle M, M \rangle_t := F(t)$.

Uniqueness: F, F' - two such processes $\Rightarrow$

$F - F'$ — bounded variation, martingale $\Rightarrow$

$F - F' \equiv 0$. ∎

Corollary Let T - stopping time. Then

$\langle M^T, M^T \rangle_t = \langle M, M \rangle_t^T \quad (:= \langle M, M \rangle_{\min(t,T)})$

stopped martingale — stopped quadratic variation

Proof. By OST, $(M_t^T)^2 - \langle M, M \rangle_t^T$ - martingale.

By uniqueness, $\langle M, M \rangle_t^T = \langle M^T, M^T \rangle_t$. ∎

But the theorem does not even work for Brownian Motion!

Need:

Def Local martingale:

(M_t) - $(\mathcal{F}_t)$ adapted process: $\exists$ sequence of stopping time T_n such that:

1) $T_n \uparrow$, $\lim_{n \to \infty} T_n = \infty$ a.s

2) $\forall n$, $M_t^{T_n} \mathbb{1}_{\{T_n > 0\}}$ is a uniformly integrable martingale.

Remarks:

1) Every martingale is a local martingale. ($T_n = n$).

2) Define $S_n := \min(\inf\{t : |M_t| \geq n\}, T_n)$

Then $M_t^{S_n}$ is bounded (by n), $S_n \uparrow \infty$, $S_n \leq S_{n+1}$. So equivalently can replace

2) with bounded martingales.

3) An example of local martingale which is <u>not</u> a martingale.

Let (B_t) - standard B.M. $T = \inf\{t : B_t = -1\}$, $T < \infty$ a.s.

B_t^T is a martingale. $\lim_{t\to\infty} B_t^T = -1$ a.s.

Consider $X_t = \begin{cases} B^T_{t/(1-t)}, & 0 \le t < 1 \\ -1, & 1 \le t < \infty \end{cases}$ - continuous a.s.

For $t<1$, X_t is just reparameterization of B.M, so $E(X_t) = E(X_0) = 0$. But for $t \ge 1$, $E(X_t) = -1 \ne 0$ - <u>not</u> a martingale.

let $T_n := \begin{cases} \inf\{t : X_t = n\} \\ n, \text{ if } X_t \ne n \ \forall t \text{ (i.e. } B_t \ne n \ \forall\, t \le T) \end{cases}$

Then $T_n \nearrow \infty$, since $T_n = n$ if $n > \max_{t\le T} B_t$. So for a.s. ω,

$T_n(\omega) = n$ for $n > N(\omega) := \max_{t \le T} B_t$.

So $X_t^{T_n} = B^{\min(T_n, T)}_{t/(t+1)}$ - <u>bounded</u> (by -1 and n)

So $X_t^{T_n}$ - reparameterized stopped B.M. - martingale

Thm Let (M_t) be a continuous local martingale. Then there exists unique $\langle M, M\rangle_t$ - continuous adapted increasing process with $\langle M, M\rangle_0 \equiv 0$ such that $M_t^2 - \langle M, M\rangle_t$ - continuous local martingale.

Moreover, $\forall t$, $\forall (\Delta_n)$ - sequence of partitions of $[0,t]$ with $|\Delta_n| \to 0$, we have

$$\sup_{s \le t} |T_s^{\Delta_n}(M) - \langle M, M\rangle_s| \to 0 \text{ in Probability.}$$

<u>Proof.</u> Choose $T_n \nearrow \infty$ - stopping times such that $(M_t^{T_n})$ - bounded martingales $\forall n$.

Then $\forall n$ $\exists\, A_n$ $(= \langle M^{T_n}, M^{T_n}\rangle)$ - increasing process with $A_n(0) \equiv 0$, $(M_t^{T_n})^2 - A_n(t)$ - <u>b</u>ounded martingale.

Observe: $((M_t^{T_{n+1}})^2 - A_{n+1})^{T_n} 1_{\{T_n>0\}}$ - also a martingale.
equal to $(M_t^{T_n})^2 - A_{n+1}^{T_n}(t)$. By uniqueness, $A_{n+1}^{T_n}(t) = A_n(t)$
if $T_n > t$.
Define $\langle M, M\rangle_t = A_n(t)$ on the set $\{t < T_n\}$.
Then $(M^{T_n})^2 - \langle M, M\rangle^{T_n}$ is a bounded martingale.
$= (M^2 - \langle M, M\rangle)^{T_n}$. So $(M^2 - \langle M, M\rangle)$ is a local martingale.

For the second part, fix $\varepsilon, \delta > 0$, $t > 0$. Then $\exists U$-stopping time:
such that $M^U 1_{\{U>0\}}$ is bounded and $P(U \le t) < \delta$
(take T_n for large n). Since $T^{\Delta}(M)$ and $\langle M, M\rangle$
are the same as U-stopped version on $[0, U]$
we have
$P(\sup_{s\le t} |T_s^{\Delta_n}(M) - \langle M, M\rangle_s| > \varepsilon) \le \delta + P(\sup_{s\le t} |T_s^{\Delta_n}(M^U) -$
$\langle M^U, M^U\rangle_s| > \varepsilon)$. The last term tends to 0
as $|\Delta_n| \to 0$.
Finally, uniqueness follows from existence of q.v.,
as before ∎

Polarization. M, N - continuous local martingales
Then $\exists$ unique process $\langle M, N\rangle_t$ of bounded variation,
$\langle M, N\rangle_0 = 0$ such that $MN - \langle M, N\rangle$ - continuous local
martingale. $P\text{-}\lim \sup_{s\le t} \left| \sum (M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}}) - \langle N, M\rangle_s \right| = 0$

Proof. Existence: $\langle M, N\rangle := \frac{1}{4}(\langle M+N, M+N\rangle - \langle M-N, M-N\rangle)$
Uniqueness: From existence of q.v., as before
Limit. Both sides satisfy polarization. ∎

Remarks. 1) If T is stopping time,
$\langle M^T, N^T\rangle = \langle M, N^T\rangle = \langle M, N\rangle^T$.

(① ⇒ ③ – obvious, ② ⇒ ③, because of P-lim property,

$\forall s \le t$ $\langle M^T, N^T \rangle_s = \langle M^T, N \rangle_s = \langle M, N \rangle_s$ on $\{T \ge t\}$

$\langle M^T, N^T \rangle_t - \langle M^T, N^T \rangle_s = \langle M^T, N \rangle_t - \langle M^T, N \rangle_s = 0$ on $\{T \le s < t\}$.

This implies that $M^T N^T - M N^T$ – local martingale.

2) $\langle M, M \rangle_t = 0 \Leftrightarrow M = \text{const}$ ($M_t = M_0$ a.s. $\forall t$).

Since $\langle M, M \rangle_t = 0 \Leftrightarrow E(M_t^2) = E(M_0^2) \Leftrightarrow E((M_t - M_0)^2) = 0 \Leftrightarrow M_t = M_0$).

Def. (H_t) is called measurable if $(\omega, t) \to H_t(\omega)$ is $\mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}_+)$ measurable ($\mathcal{B}(\mathbb{R}_+)$ – Borel σ-algebra).

Proposition Let M, N – continuous local martingales, H, K – measurable processes. Then $\forall\ t \le \infty$, a.s.

$$\int_0^t |H_s| |K_s| \, |d\langle M, N \rangle_s| \le \left(\int_0^t H_s^2 \, d\langle M, M \rangle_s \right)^{\frac{1}{2}} \left(\int_0^t K_s^2 \, d\langle N, N \rangle_s \right)^{\frac{1}{2}}$$

total variation

Proof. Need only to prove for $t < \infty$, bounded H and K, and take limits.

By changing signs of H_s and K_s only need to prove that $\left| \int_0^t H_s K_s \, d\langle M, N \rangle_s \right| \le \ldots$

The usual Cauchy-type argument.

Consider $\langle M, M \rangle_s^t + 2r \langle M, N \rangle_s^t + r^2 \langle N, N \rangle_s^t \ge 0$ $\forall r \in \mathbb{R}$

(since $= \langle M + rN, M + rN \rangle_s^t \ge 0$).

So, by computing the usual discriminant,

$$|\langle M,N\rangle_s^t| \le (\langle M,M\rangle_s^t)^{1/2} (\langle N,N\rangle_s^t)^{1/2}$$

So if we take $K = \sum K_i 1_{(t_i, t_{i+1}]}$ $\quad 0 = t_0 < t_1 < t_2 < \dots < t_n = t$ - partition.

$H = \sum H_i 1_{[t_i, t_{i+1})}$

Then by this inequality:

$$\left|\int_s^t H_s K_s \, d\langle M,N\rangle_s\right| \le \sum |H_i K_i| \, |\langle M,N\rangle_{t_i}^{t_{i+1}}| \le$$

$$\left(\sum |H_i| (\langle M,M\rangle_{t_i}^{t_{i+1}})^{1/2} \cdot |K_i| (\langle N,N\rangle_{t_i}^{t_{i+1}})^{1/2}\right) \overset{\text{Cauchy}}{\le}$$

$$\left(\sum H_i^2 \langle M,M\rangle_{t_i}^{t_{i+1}}\right)^{1/2} \left(\sum K_i^2 \langle N,N\rangle_{t_i}^{t_{i+1}}\right)^{1/2} =$$

$$\left(\int_0^t H^2 d\langle M,M\rangle\right)^{1/2} \cdot \left(\int_0^t K^2 d\langle N,N\rangle\right)^{1/2}$$

Then approximate arbitrary H, K by step functions.

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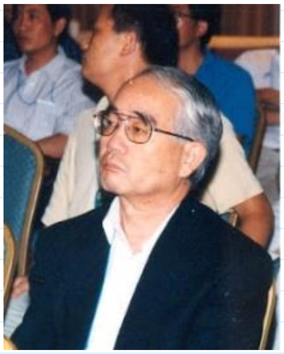

Shinzu Watanabe

Corollary (Kunita - Watanabe inequality)

$\forall p \ge 1 \quad \frac{1}{p} + \frac{1}{q} = 1$, we have

$$E\left(\int_0^\infty |H_s| |K_s| \, |d\langle M,N\rangle_s|\right) \le \left\| \left(\int_0^\infty H_s^2 d\langle M,M\rangle_s\right)^{1/2} \right\|_p \cdot$$

$$\left\| \left(\int_0^\infty K_s^2 d\langle N,N\rangle_s\right)^{1/2} \right\|_q.$$

Proof: Previous inequality + Hölder ■

Remark: We'll use it for $p = q = 2$:

$$E\left(\int_0^\infty |H_s| |K_s| \, |d\langle M,N\rangle_s|\right) \le E\left(\int_0^\infty H_s^2 \, d\langle M,M\rangle_s\right)^{1/2}$$

$$E\left(\int_0^\infty |H_s| \, |K_s| \, |d\langle M, N\rangle_s| \right) \leq E\left(\int_0^\infty H_s^2 \, d\langle M, M\rangle_s \right)^{\frac{1}{2}} E\left(\int_0^\infty K_s^2 \, d\langle N, N\rangle_s \right)^{\frac{1}{2}}$$